[unattributed results due to Greg!]

Theorem (BCR, CI)

G finite group, k a field, $\mathrm{char}(k) \mid |G|$. Then $\exists$ isom. of lattices:

$$\mathrm{Thick}^{\otimes}(D^b(kG)) \cong \left\{ \begin{array}{l} \text{specializ. closed subsets} \\ \text{of } \mathrm{Spec}^h H^*(G,k) \end{array} \right\}$$

- Saw as an exercise:

  "Building" & "realization" $\Longrightarrow$ the above bijection.

- Sebastian explained "building":

  If $V_G(M) \subseteq V_G(N) \Longrightarrow \mathrm{thick}^{\otimes}(M) \subseteq \mathrm{thick}^{\otimes}(N)$.

- Ben explained "realization", i.e.:

  Given $V \subseteq \mathrm{Spec}^h H^*(G,k)$ closed $\Rightarrow \exists\ M \in D^b(kG)$ s.t. $V_G(M) = V$.

E elem. abelian p-group of rank r

- Jesse: $\mathrm{Thick}\, D^f(kE) \cong \left\{ \begin{array}{l} \underline{\text{conical}} \text{ spec. closed} \\ \text{subsets of } \mathrm{Spec}\, H^*(E,k) \end{array} \right\}$

  Rmk: $\cong \left\{ \begin{array}{l} \text{spec. closed subsets of} \\ \mathrm{Spec}^h H^*(E,k) \end{array} \right\}$

contain origin & closed under scalar multipl.

idea: putting a grading on the ring $\cong$ remembering that $k^{\times} \curvearrowright$ affine space

affine space

homogeneous spectrum

Wanna talk a little about a part of the paper we have not discussed, namely:

Complete intersections (e.g. kE for E el.ab p-gp)

(Q, m, k) regular local ring

includes noetherian + finite global dimension

not really nec.

- A sequence $(q_1, \ldots, q_c)$ is Q-regular if each q_i is a non-zero divisor on $Q/(q_1, \ldots, q_{i-1})$ (also: non-unit!)
- $R := Q/(q_1, \ldots, q_c)$ is a complete intersection ring if $(q_1, \ldots, q_c)$ is a regular sequence.
- Hyp: $(\underline{z}) \subseteq m^2$ (out of laziness)
- Geometrically: each time we intersect with a hypersurface, the dim. drops precisely by one!
- In comm. algebra: $\dim R = \dim Q - c$. (Krull-dimension)

say: each step $-/q_i$, we are getting the worst ... ?

Example: $Q = \mathbb{F}_p[x_1, \ldots, x_r]$ (or k any instead of $\mathbb{F}_p$)

$\underline{q} = (x_1^p, \ldots, x_r^p) \rightsquigarrow R = \mathbb{F}_p E$

Theorem

Let R be a zero-dim. complete intersection, i.e.: $Q/(q_1, \dots, q_d)$, $\underline{q}$ the regular seq. has length $= d := \dim Q$. Then:

$$\text{Thick } D^b(R) \cong \left\{ \begin{array}{l} \text{spec. closed subsets} \\ \text{of } \operatorname{Spec}^h k[\theta_1, \dots, \theta_d] \end{array} \right\}$$

where $|\theta_i| \in \{1,2\}$, $k = \cancel{R}/m$ the residue field.

This generalizes directly our classif. for elementary abelian p-groups.

Now: Could look at complete intersections of higher ~~dim~~ dimension!

C.i.'s more generally

(Q, m, k) still regular local, $(q_1, \dots, q_c) \subseteq m^2$ as before, $\{q\}$ regular sequence.

Def: R has an isolated singularity if $\forall\, p \in \operatorname{Spec} R \setminus \{n\}$, R_p has finite global dimension ($n :=$ ~~R~~ the max. ideal of R)

Say: $\dim R =: \dim Q - c$, $c := \operatorname{codim} R$

Theorem Let $S := k[\theta_1, \dots, \theta_c]$, $|\theta_i| = 2$.

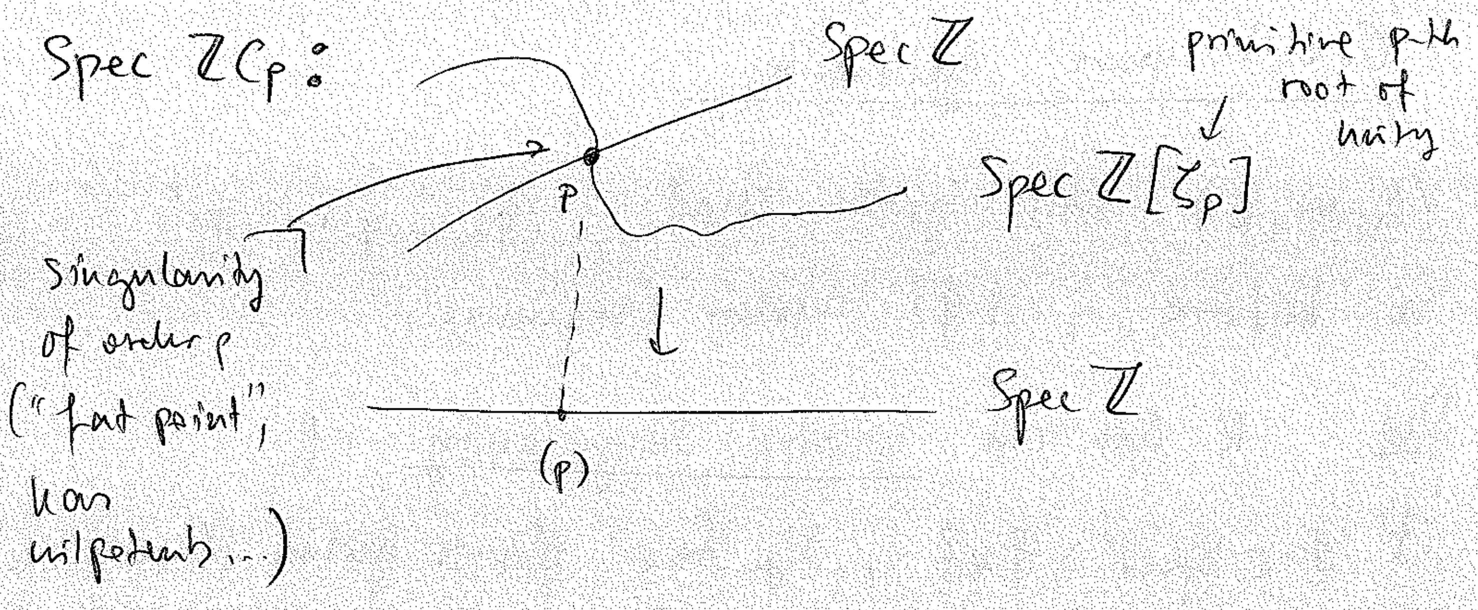
$$\boxed{\operatorname{Thick} D^b(R) \cong \left\{ \begin{array}{l} (V, W) \subseteq \operatorname{Spec}^h S \times \operatorname{Spec} R \quad \text{s.t.:} \\ V, W \text{ spe-closed} \ ; \ V \neq \emptyset \Rightarrow W \neq \emptyset \end{array} \right\}}$$

Example: (a non-local one... in fact, it works the same, as long as $\exists$! singularity)

Consider: $\mathbb{Z}C_p \cong \dfrac{\mathbb{Z}[x]}{(x^p - 1)}$ $\begin{array}{l} \leftarrow \text{regular} \\ \leftarrow \text{a non-zero div.} \end{array}$

$\rightsquigarrow$ a compl. intersection by one equation, i.e.: a hypersurface.

It has a unique singular point, namely: $(p, x-1)$. Picture:



Theorem tells us:

$$\mathrm{Thick}\, D^b(\mathbb{Z}C_p) \cong \left\{ \begin{array}{c} \text{spc-closed subsets} \\ \text{of } \operatorname{Spec} \mathbb{Z}C_p \end{array} \right\} \amalg 2$$

as lattice

i.e.: 2 possibilities for a pair (V,W)

here $S = k[\theta]$, $|\theta| = 2$ $\rightsquigarrow$ 2 possib. for V:

$\rightsquigarrow$ (ϕ, W), $((\theta), W)$. We have:

$(\emptyset, \emptyset) \longleftrightarrow 0$

$((\theta), \operatorname{Spec} \mathbb{Z}C_p) \longleftrightarrow D^b(\mathbb{Z}C_p)$ the whole cat

$(\emptyset, \operatorname{Spec} \mathbb{Z}C_p) \longleftrightarrow D^{\mathrm{perf}}(\mathbb{Z}C_p)$

other possibilities: they interpolate in between!

More general result.

Theorem Q regular, $q \in Q$ a regular element $R := Q/(q)$, then we have the following iso, where: $\mathrm{Sing}(R) := \{ \mathfrak{p} \in \operatorname{Spec}(R) \mid R_{\mathfrak{p}} \text{ not regular} \}$:

$$\mathrm{Thick}(D^b(R)) \cong \left\{ (V,W) \in \mathrm{Sing}\, R \times \operatorname{Spec} R \;\middle|\; \begin{array}{l} V, W \text{ spc-closed} \\ \text{and } V \subseteq W \end{array} \right\}$$

Assignment:

$$M \longmapsto \operatorname{supp} M = \Big(\underbrace{\{ \mathfrak{p} \mid M_{\mathfrak{p}} \text{ not perfect} \}}_{\substack{\text{singular locus} \\ \text{of the complex}}}, \underbrace{\{ \mathfrak{p} \mid M_{\mathfrak{p}} \neq 0 \}}_{\text{usual support}} \Big)$$